Design of Quadrature Mirror Hilbert Transformers with Perfect Reconstruction

Chimin Tsai
Department of Electrical Engineering, Chung Hua University, Hsin Chu, Taiwan
Tel: +886-3-5186037; Fax: +886-3-5186436; E-mail: tsai@chu.edu.tw

ABSTRACT

Applying a \( \pi/2 \) frequency shift to the original quadrature mirror filter bank (QMFBB), \( H_f(z) = H_o(-z) \), yields an equivalent system with complex half-band filters \( H_o(-zj) \) and \( H_f(zj) \). Since their coefficients are complex conjugate to each other, one can be removed. \( H_o(zj) \) can be written as a combination of an identity system and a Hilbert transformer, hence the resulting system is called the quadrature mirror Hilbert transformer (QMHT). However, a QMFBB with perfect reconstruction requires \( H_f(z) = z^{-N}H_o(-z^{-1}) \), which equality can be realized by the design of a zero phase half-band filter \( P(z) = H_o(z)H_o(-z^{-1}) \). The related QMHT is obtained by substituting \( z - jz \) into \( P(z) \). This work establishes the relationship between QMFBB and QMHT for systems with perfect reconstruction. For a given QMFBB, filter specifications and the design procedure for the QMHT are investigated.

Keywords—Quadrature mirror filter bank, perfect reconstruction, complex half-band filter, Hilbert transformer.

1. Introduction

The structure of a two-channel quadrature mirror filter bank (QMFBB) is presented in Fig. 1. \( H_o(z) \) is usually assumed to be a lowpass filter and \( H_f(z) \) to be a highpass filter. After decimation and interpolation, the signal is reconstructed by synthesis filters \( G_o(z) \) and \( H_f(z) \). It is known that the alias-free system can be designed without amplitude and/or phase distortion [1]. Instead of lowpass/highpass decomposition by \( H_o(z) \) and \( H_f(z) \), a \( \pi/2 \) frequency shift results in positive/negative frequency separation by complex half-band filters \( H_o(-zj) \) and \( H_f(zj) \). Such filters produce analytic output signals, and may be expressed as a combination of an identity system and a Hilbert transformer. This system is therefore referred to as the quadrature mirror Hilbert transformer (QMHT) [2], [3]. This work studies the filter specifications and design procedure for QMFBB with perfect reconstruction.

Section 2 discusses the original (or the earliest) QMFBB. As expected, the distortion transfer function of the associated QMHT has linear phase in the FIR design, and is allpass in the IIR design. The case of perfect reconstruction is considered in Section 3, where the relationship between QMFBB and QMHT, and the design procedure are investigated. Specifications and a design example are given in Section 4.

![Diagram](image1)

Figure 1. The two-channel QMFBB.

2. From QMFBB to QMHT

For the two-channel QMFBB in Fig. 1, the input-output relation is given by [1]

\[
\hat{x}(z) = \frac{1}{2} [H_o(z)G_o(z) + H_f(z)G_f(z)]x(z)
+ \frac{1}{2} [H_o(-z)G_o(z) + H_f(-z)G_f(z)]x(-z).
\]  

(1)

Clearly, the aliasing is removed by choosing

\[ G_o(z) = H_f(-z), \quad G_f(z) = -H_o(-z). \]  

(2)

Accordingly, the reconstructed signal \( \hat{x}(n) \) has a z-transform of the form \( \hat{x}(z) = T(z)x(z) \), in which

\[ T(z) = \frac{1}{2} [H_o(z)H_f(-z) - H_f(z)H_o(-z)] \]  

(3)

is called the distortion transfer function of the alias-free system.

In the original QMFBB, the analysis filters are related as

\[ H_f(z) = H_o(-z). \]  

(4)

Thus the system is completely determined by a single filter \( H_o(z) \). When the filter is represented in terms of polyphase components,

\[ H_o(z) = E_0(z^2) + z^{-1}E_1(z^2), \]  

(5)

the distortion transfer function in (3) is rewritten as

\[ T(z) = 2z^{-1}E_0(z^2)E_1(z^2). \]  

(6)

In linear phase FIR design, \( H_o(z) \) can be restricted as a half-band filter

\[ H_o(z) + H_o(-z) = 1, \]  

(7)
in which the zero phase is assumed for simplicity. Therefore \( H_o(z) \) is expressed as

\[
H_o(z) = \frac{1}{2} \left[ 1 + z^{-1} E(z^2) \right],
\]

where \( E(z) \) is a wide-band type II linear phase FIR filter. However, the odd-order elliptic IIR filter satisfying power-symmetry condition,

\[
H_o(z)H_o(z^{-1}) + H_o(-z)H_o(-z^{-1}) = 1,
\]

can be expressed by

\[
H_o(z) = \frac{1}{2} \left[ A_o(z) + z^{-1} A_1(z) \right],
\]

where \( A_o(z) \) and \( A_1(z) \) are allpass IIR filters. Since (8) and (10) are expressed in polyphase components, the distortion transfer function has linear phase in the FIR design, and is allpass in the IIR design.

Results of the original QMFB can be extended to complex coefficient digital filters. Let \( z \rightarrow jz \), or equivalently, \( \omega \rightarrow \omega + \pi/2 \), the analysis filters \( H_o(jz) \) and \( H_i(jz) \) decompose the input into positive and negative frequency analytic signals. In this work, input signal \( x(n) \) and the filter coefficients in Fig. 1 are assumed to be real. Therefore from (4) and (5), the filter coefficients of \( H_o(jz) \) and \( H_i(jz) \) are complex conjugates to each other. With a real input, one channel can be removed [2], [3].

Consider the one-channel system in Fig. 2. The complex half-band filter \( H(z) \) is defined as [4], [5]

\[
H(z) = 2H_o(-jz).
\]

In the frequency domain, the relation is

\[
H(e^{j\omega}) = 2H_o(e^{j(\omega + \pi/2)}).
\]

Suppose \( H_o(z) \) is an ideal half-band lowpass filter. Then the associated ideal complex half-band filter can be decomposed into two real coefficient filters

\[
H(z) = H_r(z) + jH_i(z),
\]

in which

\[
H_r(z) = 1
\]

is an identity system and

\[
H_i(z) = \begin{cases} -j, & 0 < \omega < \pi, \\ j, & -\pi < \omega < 0 \end{cases}
\]

is the ideal Hilbert transformer. For this reason, the system is also called the quadrature mirror Hilbert transformer (QMHT).

Analyzing the effects of the decimator and the interpolator in Fig. 2 reveals that

\[
\hat{x}(z) = \frac{1}{2} \left[ H(z)X(z) + H(-z)X(-z) \right] G^*(z^*).
\]

For a real input signal, \( X(z) = X^*(z^*) \), and the real part of the output signal \( \hat{x}(n) = \text{Re}\{\hat{x}(n)\} \) has a z-transform in the form

\[
\hat{x}(z) = \frac{1}{2} \left[ \hat{x}_r(z) + \hat{x}_i(z) \right] G^*(z^*)
\]

\[
= \frac{1}{4} \left[ H(z)G^*(z^*) + H^*(z^*)G(z) \right] X(z)
\]

\[
+ \frac{1}{4} \left[ H(-z)G^*(z^*) + H^*(z^*)G(z) \right] X(-z).
\]

Choosing

\[
G(z) = -jH(-z)
\]

removes the aliasing.

Similar to (13), the complex synthesis filter \( G^*(z^*) \) in Fig. 2 can be written as a combination of two real coefficient filters

\[
G^*(z^*) = G_r(z) - jG_i(z).
\]

In this case, the alias-free condition in (18) becomes that in (2) with the subscripts “0” and “1” replaced by “r” and “i”, respectively. Likewise, when the real part of the output is considered, Fig. 2 becomes Fig. 1 but with the same subscript changes. The structure is redrawn in Fig. 3 with a caption to emphasize that \( H(z) \) is a complex half-band filter, and consequently, \( H_r(z) / H_r(z) \) is a Hilbert transformer. If the QMHT is obtained from the original QMFB as in (4), then the polyphase representation in (5) is substituted into (11), yielding

\[
H(z) = 2E_0(-z^2) + j2z^{-1}E_1(-z^2).
\]

A comparison with (13) clearly indicates that

\[
H_r(z) = 2E_0(-z^2), \quad H_i(z) = 2z^{-1}E_1(-z^2).
\]

Equation (3) with the subscripts properly replaced shows that the distortion transfer function of the QMHT is

\[
T(z) = -4z^{-1}E_0(-z^2)E_1(-z^2).
\]

Clearly, this distortion transfer function is linear phase for the FIR design in (8) and allpass for the IIR design in (10). Moreover, its magnitude is twice of the shifted version of the original QMFB as in (6).

\[
\begin{align*}
H(z) & \quad \rightarrow \quad H_r(z) \quad \rightarrow \quad 12 \quad \rightarrow \quad \hat{x}(n) \\
H(z) & \quad \rightarrow \quad H_i(z) \quad \rightarrow \quad 12 \quad \rightarrow \quad \hat{x}(n)
\end{align*}
\]

Figure 3. The QMHT.
3. Perfect Reconstruction

In view of (3), the input signal in Fig. 1 can be reconstructed perfectly within a scaling factor by forcing

\[ H_0(z)H_1(-z) - H_0(-z)H_1(z) = 0. \]  
(23)

It is known that no useful system with perfect reconstruction exists under the constraint of the original QMFB in (4). In a practical design, perfect reconstruction is achieved by using causal FIR filters, which are related as [1]

\[ H_1(z) = z^{-N} H_0(-z^{-1}). \]  
(24)

in which the order of the filter, \( N \), is odd. Substituting (24) into (3) gives

\[ T(z) = -\frac{1}{2} z^{-N} [H_0(z)H_0(-z^{-1}) + H_0(-z)H_0(-z^{-1})]. \]  
(25)

Instead of (23), a practical design requires that \( H_0(z) \) satisfies the power-symmetry condition given in (9). Consequently, the resulting distortion transfer function is

\[ T(z) = -\frac{1}{2} z^{-N} \]. \]  

The design of the two-channel QMFB with perfect reconstruction is divided into two steps. First, design a zero phase half-band FIR filter \( P(z) \) satisfies

\[ P(z) + P(-z) = 1 \]  
(26)

with

\[ P(z) = H_0(z)H_0(-z^{-1}). \]  
(27)

Second, factor \( P(z) \) into \( H_0(z) \) and \( H_0(-z^{-1}). \) The common choice of \( H_0(z) \) is the one with the minimum phase.

Substituting \( z = -j\omega \) in (24) reveals that \( H_0(-j\omega) \) and \( H_1(-j\omega) \) are not conjugate related filters. In this case, one complex channel cannot be discarded while the perfect reconstruction property is retained. Therefore the procedure for designing a related QMHT must be modified. From the analysis of the one-channel complex QMFB in Fig. 2, it is observed that under the alias-free condition in (18), the signal in (17) is perfectly reconstructed by forcing [2]

\[ H(z)G^*(z^*) + H^*(z)G(z) = 4. \]  
(28)

This requirement can be met by substituting \( z = -j\omega \) into (26). Let

\[ Q(z) = P(-j\omega). \]  
(29)

it can be shown that (26) is equivalent to

\[ Q(z) + Q^*(z^*) = 1 \]  
(30)

as the zero phase FIR filter \( P(z) \) is in the form of (8). Comparing (28) and (30) suggests that a QMHT system with perfect reconstruction is obtained by choosing

\[ 4Q(z) = H(z)G^*(z^*). \]  
(31)

The power-complementary condition implies that

\[ 0 \leq P(e^{j\omega}) = |H_0(e^{j\omega})|^2 \leq 1, \]  
(32)

and consequently, any unit circle zero of \( P(z) \) and \( Q(z) \) must be double zero. Under the alias-free condition in (18), the product filter \( 4Q(z) \) is factored according to (31) such that

\[ |H(e^{j\omega})| = |G(e^{j\omega})| = 2Q^{1/2}(e^{j\omega}). \]  
(33)

Because of the frequency shift introduced by \( z = -j\omega \), a comparison of (32) and (33) shows that the \( H(z) \) in QMHT and the \( H_0(z) \) in QMFB are related as

\[ |H(e^{j\omega})| = 2|H_0(e^{j(\omega-\pi/2)})|. \]  
(34)

Unlike the previous case in (12), the phase relationship is not specified in (34). Furthermore, when the complex filters are realized by real coefficient digital filters as shown in Fig. 3, the requirement for perfect reconstruction given by (28) becomes

\[ H_1(z)H_1(-z) - H_1(-z)H_1(z) = 2. \]  
(35)

This equation is clearly of the same form as (23) but the magnitude is doubled.

4. Design of QMHT with PR

The actual design is focused on a linear phase half-band FIR filter

\[ P(z) = P(-z) = z^{-N}, \]  
(36)

in which

\[ P(z) = \frac{1}{2} \left(z^{-N} + E(z^2)\right). \]  
(37)

Consider a wide-band FIR filter \( E'(z) \) of odd-order \( N \)

\[ \left|E'(e^{j\omega})\right| \]

\[ \left|\hat{p}(e^{j\omega})\right| \]

Figure 4. Filter specifications.
A typical example of the magnitude response is given in Fig. 4, where the ripple is \( \delta' \). Let

\[
E(z) = \frac{1}{1 + \delta'} E'(z)
\]

(38)
and substitute it into (37). The ripple of \( P(z) \) is

\[
\delta = \frac{\delta'}{1 + \delta'},
\]

(39)
as shown in Fig. 4.

MATLAB was used to design \( E'(z) \) with \( N = 9 \), \( \omega_p = 0.85\pi \), \( \omega_s = 0.9\pi \), and \( \delta'/\delta_s = 1/40 \). In result, we have \( \delta' = 0.0545 \) and \( \delta = 0.0517 \). If \( (N-1)/4 \) is an integer, for example \( N = 9 \) as above, then (36) is multiplied by \( -j \) and rewritten as

\[
Q(z) + Q^*(-z^*) = z^{-N},
\]

(40)
in which \( Q(z) \) is redefined by

\[
Q(z) = -jP(-jz).
\]

(41)
From the zeros of \( Q(z) \) and the constraints given in (18) and (33), it can be shown that the zeros of \( H(z) \) occur in negative reciprocal [2], as displayed in Fig. 5. Factored according to (31), the magnitude response of \( H(z) \) is given in Fig. 6, which shows that it is indeed a complex half-band filter. Moreover, the phase difference between \( H_r(z) \) and \( H_i(z) \) reveals the characteristic of the Hilbert transformer. The list of the impulse response \( h(n) \) is given in Table I. It is observed that the real part \( h_r(n) \) and the imaginary part \( h_i(n) \) satisfy the constraint for perfect reconstruction as in (24).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \text{impulse response} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.3054 + j 0.0000</td>
</tr>
<tr>
<td>1</td>
<td>0.1271 + j 0.0006</td>
</tr>
<tr>
<td>2</td>
<td>0.2121 + j 0.0714</td>
</tr>
<tr>
<td>3</td>
<td>0.3277 + j 0.0303</td>
</tr>
<tr>
<td>4</td>
<td>0.3762 + j 0.7686</td>
</tr>
<tr>
<td>5</td>
<td>-0.7686 + j 0.3762</td>
</tr>
<tr>
<td>6</td>
<td>0.0303 - j 0.3277</td>
</tr>
<tr>
<td>7</td>
<td>-0.0714 + j 0.2121</td>
</tr>
<tr>
<td>8</td>
<td>0.0006 - j 0.1271</td>
</tr>
<tr>
<td>9</td>
<td>-0.0000 + j 0.3054</td>
</tr>
</tbody>
</table>

### 5. Conclusion

This work established the relationship between the complex half-band filter in QMHT and the real half-band filter in QMFB for perfect reconstruction. Filter specifications and design procedure were outlined, and verified using an example.

### REFERENCES


