Trees

Rosen 5th ed., Chap. 9

What are Trees?

- Connected graphs that contain no simple circuit

No more a tree if this edge is added
Unique Path Property

• An undirected graph is a tree iff there is a unique path between any two vertices.

![Diagram of a tree](image1)

Rooted Tree

• One vertex has been designated as the root and every edge is directed away from the root.

![Diagram of a rooted tree](image2)

- $u$ is the parent of $v$
- $v$ is the child of $u$
- $v$, $w$, $x$ are siblings.
- $f$ is an ancestor of $h$.
- $h$ is a descendant of $f$. 
Leaf and Internal Vertices

- Leaf: a vertex that has no children.
- International vertices: non-leaf nodes.

Which are leaf vertices? Which are internal vertices?

Subtree

- What is the subtree rooted at $u$?
- What is the rooted tree (with root $f$)?
**m-ary Tree**

- A rooted tree is an *m*-ary tree if every internal node has no more than *m* children.

![Diagram of a 3-ary tree](image)

A 3-ary tree. Not a binary tree.

**Full *m*-ary Tree**

- A rooted tree is a full *m*-ary tree if every internal node has exactly *m* children.

![Diagram of a full binary tree](image)

A full binary tree.
Binary Tree

- In an ordered rooted tree, the children of each internal vertex are ordered.
- In an ordered binary tree,

![Diagram of a binary tree]

Properties of Trees

- A tree with $n$ vertices has $n - 1$ edges.
  - Otherwise circuits will be created
- A full $m$-ary tree with $i$ internal vertices contains $n = mi + 1$ vertices.

![Diagram of a tree]

$\begin{align*}
  m &= 2 \\
  i &= 3 \\
  n &= 7
\end{align*}$

Every vertex (except the root) is the child of an internal vertex.

總共 $mi$ 個這種節點
More Properties of Trees

- A full $m$-ary tree with $n$ vertices has $i = (n-1)/m$ internal vertices and $l = [(m-1)n+1]/m$ leaves.

Proof:

full $m$-ary tree, so

$n = mi + 1 \Rightarrow i = (n-1)/m$

also

$n = i + l \Rightarrow l = [(m-1)n+1]/m$

Level of Vertices

- Level of a vertex: the length of the unique path from root to it.
- Height of a tree: the maximum level.
Balanced Tree

- A rooted $m$-ary tree of height $h$ is balanced if all leaves are at levels $h$ or $h - 1$.

Properties on Height

- There are at most $m^h$ leaves in an $m$-ary tree of height $h$.

Proof (by strong induction on $h$):

$h = 1$: $m$-ary trees of height 1 $\Rightarrow$ at most $m$ leaves $\Rightarrow$ true.

Assume that the result is true for all $m$-ary trees of height less than $h$.

$\Rightarrow$ any $m$-ary tree of height $k < h$ has $\leq m^k$ leaves.
Proof (Cont)

Let $T$ be an $m$-ary tree of height $h$

- 1st subtree of height $\leq h-1$
- 2nd subtree of height $\leq h-1$
- $m$th subtree of height $\leq h-1$

Each has $\leq m^{h-1}$ leaves.

There are at most $m$ such subtree in $T$.

$\Rightarrow T$ has at most $m \times m^{h-1}$ leaves.

$\Rightarrow T$ has at most $m^h$ leaves.