Recurrence Relations

Rosen 5th ed., §6.1-6.2

§6.1: Recurrence Relations

• A recurrence relation (R.R., or just recurrence) for a sequence \( \{a_n\} \) is an equation that expresses \( a_n \) in terms of one or more previous elements \( a_0, \ldots, a_{n-1} \) of the sequence, for all \( n \geq n_0 \).
  – A recursive definition, without the base cases.

• Example

\[
a_n = 2a_{n-1} - a_{n-2} \quad (n \geq 2)
\]
Solutions of a Recurrence

- A particular sequence (described non-recursively) is said to solve the given recurrence relation if it is consistent with the definition of the recurrence.
  - A given recurrence relation may have many solutions.
- Example: the solutions of $a_n = 2a_{n-1} - a_{n-2}$ ($n \geq 2$).
  
  \[
  a_n = 3n? \quad Yes \quad 2[3(n-1)] - 3(n-2) = 3n = a_n \\
  a_n = 2^n? \quad No \quad 2(2^{n-1}) - 2^{n-2} \neq 2^n = a_n \\
  a_n = 5? \quad Yes \quad 2 \cdot 5 - 5 = 5 = a_n 
  \]

Example Applications

- Recurrence relation for growth of a bank account with $P\%$ interest per given period
- $M_n$: the amount of account after $n$ years
  
  \[
  M_n = M_{n-1} + (P/100)M_{n-1} 
  \]
- Example: $M_0 = 10,000$; $P=11$
  
  \[
  M_0 = 10000 \\
  M_1 = 10000 + 11/100 \times 10000 = 11100 \\
  M_2 = 11100 + 11/100 \times 11100 = 12321 \\
  M_3 = \ldots 
  \]
Solving Compound Interest RR

- \( M_n = M_{n-1} + (P/100)M_{n-1} \)
  - \( = (1 + P/100) M_{n-1} \)
  - \( = r M_{n-1} \) (let \( r = 1 + P/100 \))
  - \( = r (r M_{n-2}) \)
  - \( = r \cdot r (r M_{n-3}) \) ...and so on to...
  - \( = r^n M_0 \)

Grow of a Population

- Growth of a population in which each organism yields 1 new one every period starting 2 periods after its birth.
  \[ P_n = P_{n-1} + P_{n-2} \] (Fibonacci relation)
Tower of Hanoi Example

• Problem: Get all disks from peg 1 to peg 2.
  – Only move 1 disk at a time.
  – Never place a larger disk on a smaller one.

Hanoi Recurrence Relation

• Let $H_n = \# \text{ moves for a stack of } n \text{ disks.}$
• Optimal strategy:
  – Move top $n-1$ disks to spare peg. ($H_{n-1}$ moves)
  – Move bottom disk. (1 move)
  – Move top $n-1$ to bottom disk. ($H_{n-1}$ moves)
• Note: $H_n = 2H_{n-1} + 1$
Solving Tower of Hanoi RR

\[ H_n = 2 \ H_{n-1} + 1 \]
\[ = 2 \ (2 \ H_{n-2} + 1) + 1 = 2^2 \ H_{n-2} + 2 + 1 \]
\[ = 2^2(2 \ H_{n-3} + 1) + 2 + 1 = 2^3 \ H_{n-3} + 2^2 + 2 + 1 \]
\[ \vdots \]
\[ = 2^{n-1} \ H_1 + 2^{n-2} + \ldots + 2 + 1 \]
\[ = 2^{n-1} + 2^{n-2} + \ldots + 2 + 1 \quad (\text{since } H_1 = 1) \]
\[ = \sum_{i=0}^{n-1} 2^i \]
\[ = 2^n - 1 \]

§6.2: Solving Recurrences

- A **linear homogeneous recurrence of degree** \( k \) **with constant coefficients** ("k-LiHoReCoCo") is a recurrence of the form

\[ a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k}, \]

where the \( c_i \) are all real, and \( c_k \neq 0 \).

- The solution is uniquely determined if \( k \) initial conditions \( a_0 \ldots a_{k-1} \) are provided.
LiHoReCoCo Examples

- $M_n = (1.11) M_{n-1}$
  - LiHoReCoCo of degree 1
- $f_n = f_{n-1} + f_{n-2}$
  - LiHoReCoCo of degree 2
- $a_n = a_{n-5}$
  - LiHoReCoCo of degree 5
- $a_n = a_{n-1} + a_{n-2}^2$: not linear
- $H_n = 2H_{n-1} + 1$: not homogeneous

Solving LiHoReCoCos

- Basic idea: Look for solutions of the form $a_n = r^n$, where $r$ is a constant.
- This requires the characteristic equation:
  
  \[ r^n = c_1 r^{n-1} + \ldots + c_k r^{n-k}, \text{ i.e.,} \]
  
  \[ r^k - c_1 r^{k-1} - \ldots - c_k = 0 \]

- The solutions (characteristic roots) can yield an explicit formula for the sequence.
Solving 2-LiHoReCoCos

• Consider an arbitrary 2-LiHoReCoCo:
  \[ a_n = c_1 a_{n-1} + c_2 a_{n-2} \]

• It has the characteristic equation (C.E.):
  \[ r^2 - c_1 r - c_2 = 0 \]

• Thm. 1: If this CE has 2 roots \( r_1 \neq r_2 \), then
  \[ a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \] for \( n \geq 0 \)
  for some constants \( \alpha_1, \alpha_2 \).

• The values of \( \alpha_1 \) and \( \alpha_2 \) can be obtained
  with initial conditions

Example

• Solve the recurrence \( a_n = a_{n-1} + 2a_{n-2} \) given the initial conditions \( a_0 = 2, \ a_1 = 7 \).

• Solution: Use theorem 1
  – \( c_1 = 1, \ c_2 = 2 \)
  – Characteristic equation:
    \[ r^2 - r - 2 = 0 \]
  – Solutions: \( r^2 - r - 2 = 0 \)
    \( \Rightarrow (r - 2)(r + 1) = 0 \), so \( r = 2 \) or \( r = -1 \).
  – So \( a_n = \alpha_1 2^n + \alpha_2 (-1)^n \).
Example Continued…

• To find $\alpha_1$ and $\alpha_2$, solve the equations for the initial conditions $a_0$ and $a_1$:
  
  \[
  a_0 = 2 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot (-1)^0 \\
  a_1 = 7 = \alpha_1 \cdot 2^1 + \alpha_2 \cdot (-1)^1
  \]

  Simplifying, we have the pair of equations:
  
  \[
  2 = \alpha_1 + \alpha_2 \\
  7 = 2 \alpha_1 - \alpha_2
  \]

  which we can solve easily by substitution:
  
  \[
  \alpha_2 = 2 - \alpha_1; \quad 7 = 2 \alpha_1 - (2 - \alpha_1) = 3 \alpha_1 - 2; \\
  9 = 3 \alpha_1; \quad \alpha_1 = 3; \quad \alpha_2 = 1.
  \]

• Final answer: $a_n = 3 \cdot 2^n - (-1)^n$

\[
\text{Check: } \{a_n\} = 2, 7, 11, 25, 47, 97 \ldots
\]

The Case of Degenerate Roots

• Now, what if the C.E. $r^2 - c_1 r - c_2 = 0$ has only 1 root $r_0$? (重根)
  
  – For example, $c_1 = 2$, $c_2 = -1$

• Theorem 2: Then, $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for all $n \geq 0$, for some constants $\alpha_1$, $\alpha_2$. 
**k-LiHoReCoCos**

- Consider a $k$-LiHoReCoCo: $a_n = \sum_{i=1}^{k} c_i a_{n-i}$
- It’s C.E. is: $r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$
- **Thm.3**: If this has $k$ distinct roots $r_p$, then the solutions to the recurrence are of the form:

$$a_n = \sum_{i=1}^{k} \alpha_i r_i^n$$

for all $n \geq 0$, where the $\alpha_i$ are constants.

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**Degenerate k-LiHoReCoCos**

- Suppose there are $t$ roots $r_1, \ldots, r_t$ with multiplicities $m_1, \ldots, m_t$. Then:

$$a_n = \sum_{i=1}^{t} \left( \sum_{j=0}^{m_i-1} \alpha_{i,j} n^j \right) r_i^n$$

for all $n \geq 0$, where all the $\alpha_{i,j}$ are constants.
- Example: roots of a CE: 2, 2, 2, 5, ,5, 9

$$(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)2^n + (\alpha_{2,0} + \alpha_{2,1} n)5^n + \alpha_{3,0} 9^n$$
LiNoReCoCos

• Linear **nonhomogeneous** RRs with constant coefficients may (unlike LiHoReCoCos) contain some terms \( F(n) \) that depend *only* on \( n \) (and *not* on any \( a_i \)'s).

General form:
\[
a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} + F(n)
\]

The *associated homogeneous recurrence relation* (associated LiHoReCoCo).

Solutions of LiNoReCoCos

• A useful theorem about LiNoReCoCos:
  - If \( a_n = p(n) \) is any *particular* solution to the LiNoReCoCo
    \[
a_n = \left( \sum_{i=1}^{k} c_i a_{n-i} \right) + F(n)
    \]
  - Then *all* its solutions are of the form:
    \[
a_n = p(n) + h(n),
    \]
    where \( a_n = h(n) \) is any solution to the associated LiHoReCoCo
    \[
a_n = \left( \sum_{i=1}^{k} c_i a_{n-i} \right)
    \]
Example

• Find all solutions to $a_n = 3a_{n-1} + 2n$. Which solution has $a_1 = 3$?
  – Notice this is a \(1\)-\(\text{LiNoReCoCo}\). Its associated \(1\)-\(\text{LiHoReCoCo}\) is $a_n = 3a_{n-1}$, whose solutions are all of the form $a_n = \alpha 3^n$. Thus the solutions to the original problem are all of the form $a_n = p(n) + \alpha 3^n$. So, all we need to do is find one $p(n)$ that works.

Trial Solutions

• If the extra terms $F(n)$ are a degree-$t$ polynomial in $n$, you should try a degree-$t$ polynomial as the particular solution $p(n)$.
• This case: $F(n)$ is linear so try $a_n = cn + d$.
  \[
  cn + d = 3(c(n-1)+d) + 2n \quad \text{(for all $n$)}
  \]
  \[
  (-2c-2)n + (3c-2d) = 0 \quad \text{(collect terms)}
  \]
  So $c = -1$ and $d = -3/2$.
  So $a_n = -n - 3/2$ is a solution. ($p(n)$ found)
• Check: $a_{n \geq 1} = \{-5/2, -7/2, -9/2, \ldots \}$
Finding a Desired Solution

• From the previous, we know that all general solutions to our example are of the form:

\[ a_n = \rho(n) - l n - \frac{3}{2} + \alpha \cdot 3^n \]

Solve this for \( \alpha \) for the given case, \( a_1 = 3 \):

\[ 3 = -1 - \frac{3}{2} + \alpha \cdot 3^1 \]

\[ \alpha = \frac{11}{6} \]

• The answer is \( a_n = -n - \frac{3}{2} + \left(\frac{11}{6}\right)3^n \)

Finding a Particular Solution (I)

• Given a LiNoReCoCo

\[ a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} + F(n) \]

with \( F(n) = (b_0 n^t + \ldots + b_1 n + b_0) s^n \), where \( b_0, b_1, \ldots, b_t \), and \( s \) are real numbers.

• IF \( s \) is not a root of the C.E. of the associated LiHoReCoCo, there is a particular solution of the form

\[ (p_0 n^t + \ldots + p_1 n + p_0) s^n \]
Finding a Particular Solution (||)

• Given a LiNoReCoCo
  \[ a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k} + F(n) \]
  with \( F(n) = (b_t n^t + \ldots + b_1 n + b_0) s^n \), where
  \( b_0, b_1, \ldots, b_t \), and \( s \) are real numbers.

• IF \( s \) is a root of the C.E. of the associated LiHoReCoCo and its multiplicity is \( m \), there is a particular solution of the form
  \[ n^m (p_t n^t + \ldots + p_1 n + p_0) s^n \]

Exercise

• Find all solutions of the recurrence relation
  \[ a_n = 5a_{n-1} - 6a_{n-2} + 7n \]

  ① Find the general solution of the associated LiHoReCoCo
  ② Find a particular solution of \( a_n \)
    – Is \( F(n) \) of the form \( (b_t n^t + \ldots + b_1 n + b_0) s^n \)?
    – Is \( s \) a root of the C.E. of the associated LiHoReCoCo?
  ③ Combine these two solutions.